

# Scalar field propagation in the $\phi^4$ $\kappa$ -Minkowski model

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**ABSTRACT:** In this article we use the noncommutative (NC)  $\kappa$ -Minkowski  $\phi^4$  model based on the  $\kappa$ -deformed star product,  $(\star_h)$ . The action is modified by expanding up to linear order in the  $\kappa$ -deformation parameter  $a$ , producing an effective model on commutative spacetime. For the computation of the tadpole diagram contributions to the scalar field propagation/self-energy, we anticipate that statistics on the  $\kappa$ -Minkowski is specifically  $\kappa$ -deformed. Thus our prescription in fact represents *hybrid* approach between standard quantum field theory (QFT) and NCQFT on the  $\kappa$ -deformed Minkowski spacetime, resulting in a  $\kappa$ -effective model. The propagation is analyzed in the framework of the two-point Green's function for low, intermediate, and for the Planckian propagation energies, respectively. Semiclassical/hybrid behavior of the first order quantum correction do show up due to the  $\kappa$ -deformed momentum conservation law. For low energies, the dependence of the tadpole contribution on the deformation parameter  $a$  drops out completely, while for Planckian energies, it tends to a fixed finite value. The mass term of the scalar field is shifted and these shifts are very different at different propagation energies. At the Planckian energies we obtain the direction dependent  $\kappa$ -modified dispersion relations. Thus our  $\kappa$ -effective model for the massive scalar field shows a birefringence effect.

**KEYWORDS:** kappa-deformed space, noncommutative quantum field theory.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Mathematical preliminaries of <math>\kappa</math>-deformed Minkowski spacetime</b>	<b>2</b>
2.1 $\kappa$ -deformed algebra	2
2.2 Hopf algebra and star product	4
<b>3. Modified <math>\kappa</math>-deformed scalar field action</b>	<b>5</b>
3.1 Hermitian realization of the NC $\phi^4$ action	5
3.2 Equations of motion and Noether currents of internal symmetry	6
<b>4. Perturbative study of the model two-point function</b>	<b>6</b>
4.1 Feynman rules	6
4.1.1 Feynman rules (A): standard momentum addition law	7
4.1.2 Feynman rules (B): $\kappa$ -deformed momentum addition law	8
4.2 Scalar field propagation	12
4.2.1 Tadpole diagram: standard momentum conservation	12
4.2.2 Tadpole diagram: $\kappa$ -deformed momentum conservation	14
<b>5. Discussion and conclusion</b>	<b>17</b>

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## 1. Introduction

Recently, it was established that if the  $\kappa$ -Poincaré Hopf algebra is assumed to describe physics on  $\kappa$ -Minkowski spacetime [1, 2, 3], then it is necessary to accept certain modifications in statistics obeyed by the particles. This means that  $\kappa$ -Minkowski spacetime leads to a modification of particle statistics which results in deformed oscillator algebras [4, 5, 6, 7]. A deformation of the Poincaré algebra can be performed by means of the twist operator [8, 9, 10, 11, 12] which happens to include the dilatation generator. And thus belongs to the universal enveloping algebra of the general linear algebra [13, 14, 15, 16]. This twist operator gives rise to a deformed statistics [5, 14].

Transformation from noncommutative  $\kappa$ -Minkowski to Minkowski spacetime in the case of the free field theory was described in [17], while the star product and interacting fields on  $\kappa$ -Minkowski space were treated in the same approach in [18]. We are continuing along the line where the main aim is to transcribe original NCQFT on  $\kappa$ -Minkowski to a corresponding commutative QFT on the standard Minkowski spacetime. With this in mind, we are considering the  $\kappa$ -deformation of a Minkowski spacetime whose symmetry has a undeformed Lorentz sector (classical basis [19, 20]). The noncommutative coordinates close

in a  $\kappa$ -deformed Lie algebra and additionally, form a Lie algebra together with the Lorentz generators [21, 22, 23, 24]. This deformation of the spacetime structure affects the algebra of physical fields, leading to a modification of multiplication in the corresponding universal enveloping algebra, requiring the replacement of the usual pointwise multiplication by a deformed star product, i.e. by the new star product  $\star_h$ . Next, the action is modified by truncating the initial model via expansion up to first order in the deformation parameter  $a$ , producing an effective model on commutative spacetime.

We have to stress that by such a truncation of the  $\kappa$ -deformed action we have lost the nonperturbative quantum effects like the celebrated UV/IR mixing [25], which, amongst other, connects the noncommutative field theories with Holography [26, 27] via UV and IR cutoffs, in a model independent way [26]. Resummation of the expanded action could in principle restore the nonperturbative character of the theory, like the model in [28], thus producing UV/IR mixing [29], which under such circumstances would help in determining what the UV theory might be. That is, it would help to determine the UV completeness of the theory. However, we have to admit that for our  $\kappa$ -Minkowski  $\phi^4$  model it is absolutely not clear that a resummation of higher orders in  $a_\mu$  will give a meaningful model beyond the tree level. Those are general properties of the most of NCQFT expanded/resummed in terms of the noncommutative deformation parameter. Holography and UV/IR mixing are in the literature known as possible windows to quantum gravity [26, 27, 30].

Our approach generally represents a *hybrid* approach modeling between standard quantum field theory and NCQFT on  $\kappa$ -Minkowski spacetime involving  $\kappa$ -deformed momentum conservation law [31, 32].

In the first section, we give some mathematical preliminaries including the Hopf algebra structure of the  $\kappa$ -deformed Minkowski spacetime and the star products. In the second section, we introduce the hermitian realization and the  $\kappa$ -deformed star product  $\star_h$  corresponding to this realization. The modified  $\kappa$ -deformed scalar field action based on the above notions is introduced next, and the equations of motion are derived, with the corresponding conserved currents. The properties of the  $\kappa$ -deformed action and the perturbative study of the model two-point Green's function, are discussed in the last section.

## 2. Mathematical preliminaries of $\kappa$ -deformed Minkowski spacetime

### 2.1 $\kappa$ -deformed algebra

We are considering  $\kappa$ -deformation of Minkowski spacetime whose symmetry has an undeformed Lorentz sector and whose noncommutative coordinates  $\hat{x}_\mu$ , ( $\mu = 0, 1, \dots, n-1$ ), close in a Lie algebra together with the Lorentz generators  $M_{\mu\nu}$ , ( $M_{\mu\nu} = -M_{\nu\mu}$ ),

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu), \quad (2.1)$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda}, \quad (2.2)$$

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} - i(a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}), \quad (2.3)$$

where the deformation parameter  $a_\mu$  is a constant Lorentz vector, and  $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$  defines the metric in this spacetime. The quantity  $a^2 = a_\mu a^\mu$  is Lorentz invariant

having a dimension of inverse mass squared,  $a^2 \equiv \frac{1}{\kappa^2}$ . The above algebra has all the Jacobi identities satisfied, thus forming a Lie algebra with the property that in the limit  $a_\mu \rightarrow 0$ , the commutative spacetime with the usual action of the Lorentz algebra is recovered. Throughout the paper we shall work in units  $\hbar = c = 1$ .

The symmetry of the deformed spacetime (2.1) is assumed to be described by an undeformed Poincaré algebra. Thus, in addition to Lorentz generators  $M_{\mu\nu}$ , we also introduce momenta  $p_\mu$  which transform as vectors under the Lorentz algebra,

$$[p_\mu, p_\nu] = 0, \quad [M_{\mu\nu}, p_\lambda] = \eta_{\nu\lambda} p_\mu - \eta_{\mu\lambda} p_\nu. \quad (2.4)$$

For convenience, we refer to the algebra (2.1)-(2.5) as the deformed special relativity algebra since its different realizations lead to different special relativity models with different physics encoded in the deformed dispersion relations resulting from such theories. This algebra, however, does not fix the commutation relation between  $p_\mu$  and  $\hat{x}_\nu$ . In fact, there are infinitely many possibilities for the commutation relation between  $p_\mu$  and  $\hat{x}_\nu$ , all of which are consistent with the algebra (2.1)-(2.5) in the sense that the Jacobi identities are satisfied between all generators of the algebra. In this way, we have an extended algebra, which includes the generators  $M_{\mu\nu}$ ,  $p_\mu$  and  $\hat{x}_\lambda$  and satisfies the Jacobi identities for all combinations of the generators. Particularly, the algebra generated by  $p_\mu$  and  $\hat{x}_\nu$  is a deformed Heisenberg-Weyl algebra which in this paper we take to have the following form

$$[p_\mu, \hat{x}_\nu] = -i\eta_{\mu\nu} \left( ap + \sqrt{1 + a^2 p^2} \right) + ia_\mu p_\nu. \quad (2.5)$$

This particular type of phase space noncommutativity leads to uncertainty relations of the form [33, 34]

$$\Delta x_\mu \geq \frac{\hbar}{\Delta p_\mu} + \alpha G \Delta p_\mu, \quad (2.6)$$

( $\alpha$  is a constant and  $G$  is the gravitational constant) which have been obtained from the study of string collisions at Planckian energies, i.e. so called gravity collapse of strings [34], thus manifesting its dynamical origin. The same generalized uncertainty principle emerges from considerations related to quantum gravity [33].

The algebra (2.1)-(2.5) can be realized by

$$\begin{aligned} p_\mu &= -i\partial_\mu = -i\frac{\partial}{\partial x^\mu}, \quad M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \\ \hat{x}_\mu &= x_\mu \sqrt{1 + a^2 p^2} - iM_{\mu\nu} a^\nu, \end{aligned} \quad (2.7)$$

where  $\partial_\mu$  and  $x_\nu$  are generators of the undeformed Heisenberg algebra  $[x_\mu, x_\nu] = [\partial_\mu, \partial_\nu] = 0$ ,  $[\partial_\mu, x_\nu] = \eta_{\mu\nu}$ . The phase space noncommutativity (2.5), together with  $\kappa$ -Poincaré algebra specified by relations (2.2) and (2.4) corresponds to the classical basis of the  $\kappa$ -Poincaré algebra considered in [20, 35, 36, 37].

The algebra (2.1)-(2.4), as it is defined, is not complete. With the relations (2.5), we obtain an extended algebra containing the deformed Heisenberg-Weyl as a subalgebra. In [38] the authors started from the coproduct (Hopf algebra + module algebra) and then determined the crossed commutation relation of the extended algebra (2.1)-(2.5). Our route is just opposite: we close the algebra by the crossed commutation relation and then accordingly determine the coproduct.

## 2.2 Hopf algebra and star product

The symmetry underlying  $\kappa$ -deformed Minkowski space, characterized by the commutation relations (2.1), is the deformed Poincaré symmetry which can most conveniently be described in terms of Hopf algebras. As it was manifested in relations (2.2) and (2.5), the algebraic sector of this deformed symmetry is the same as that of the undeformed Poincaré algebra. However, the coalgebraic sector is deformed and it is determined by the coproducts for translation ( $p_\mu = -i\partial_\mu$ ), rotation and the boost generators ( $M_{\mu\nu}$ ) [22, 23],

$$\Delta\partial_\mu = \partial_\mu \otimes Z^{-1} + \mathbf{1} \otimes \partial_\mu + ia_\mu(\partial_\lambda Z) \otimes \partial^\lambda - \frac{ia_\mu}{2}\square Z \otimes ia\partial, \quad (2.8)$$

$$\begin{aligned} \Delta M_{\mu\nu} = & M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \\ & + ia_\mu \left( \partial^\lambda - \frac{ia^\lambda}{2}\square \right) Z \otimes M_{\lambda\nu} - ia_\nu \left( \partial^\lambda - \frac{ia^\lambda}{2}\square \right) Z \otimes M_{\lambda\mu}, \end{aligned} \quad (2.9)$$

where  $\otimes$  denotes the tensor product. In the above expressions,  $Z$  is the shift operator, determined by  $Z^{-1} = ap + \sqrt{1 + a^2 p^2}$ ,  $Z = 1/Z^{-1}$ . The operator  $\square = \frac{2}{a^2}(1 - \sqrt{1 - a^2 \partial^2})$ , is a deformed d'Alembertian operator [21, 23], which in the limit  $a \rightarrow 0$  acquires the standard form,  $\square \rightarrow \partial^2$ , valid in undeformed Minkowski space.

The Hopf algebra in question also has well defined counits and antipodes. The antipodes for the generators of the  $\kappa$ -Poincaré Hopf algebra and the related operator  $Z$  are given by

$$S(\partial_\mu) = \left( -\partial_\mu + ia_\mu \partial^2 + \frac{1}{2}a_\mu(a\partial)\square \right) Z, \quad (2.10)$$

$$S(M_{\mu\nu}) = -M_{\mu\nu} + ia_\mu \left( \partial_\alpha - \frac{ia_\alpha}{2}\square \right) M_{\alpha\nu} - ia_\nu \left( \partial_\alpha - \frac{ia_\alpha}{2}\square \right) M_{\alpha\mu}, \quad (2.11)$$

while the counits remain trivial.

Once we have the coproduct (2.8), we can straightforwardly construct a star product between two arbitrary fields  $f$  and  $g$  of commuting coordinates [21, 22]. For the noncommutative spacetime (2.1), the star product has the following form

$$(f \star g)(x) = \lim_{\substack{u \rightarrow x \\ y \rightarrow x}} \mathcal{M} \left( e^{x^\mu(\Delta - \Delta_0)\partial_\mu} f(u) \otimes g(y) \right), \quad (2.12)$$

where  $\Delta_0\partial_\mu = \partial_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \partial_\mu$ ,  $\Delta(\partial_\mu)$  is given in (2.8) and  $\mathcal{M}$  is the multiplication map in the undeformed Hopf algebra, namely,  $\mathcal{M}(f(x) \otimes g(x)) = f(x)g(x)$  [39]. At this point we emphasize here once again that in this paper we are doing analysis based on the specific realization  $\hat{x}_\mu = x_\mu \sqrt{1 + a^2 p^2} - iM_{\mu\nu}a^\nu$  and its hermitian variant (3.1), defined in the next section. The coproducts (2.8) and (2.9) correspond to this particular type of realization. One can check that the star product (2.12) together with the coproduct (2.8) is associative.

Note that the commutator (2.1) can be written in terms of ordinary coordinates and the star product from [24] as

$$[x_\mu, x_\nu]_\star = x_\mu \star x_\nu - x_\nu \star x_\mu = i(a_\mu x_\nu - a_\nu x_\mu). \quad (2.13)$$

### 3. Modified $\kappa$ -deformed scalar field action

In this section, we present an interacting scalar field model on noncommutative spacetime whose short distance geometry is governed by the  $\kappa$ -deformed symplectic structure (2.1). In order to obtain the physical meaning of the NC  $\phi^4$  field theory, we have to introduce a complex scalar field  $\phi$  with the accompanying notion of the hermitian conjugation operation [24].

#### 3.1 Hermitian realization of the NC $\phi^4$ action

In order to obtain the hermitian action, we are necessarily forced to work with a hermitian realization represented by the operator  $\hat{x}_\mu^h$ , having the property  $(\hat{x}_\mu^h)^\dagger = \hat{x}_\mu^h$ . The hermitian operator  $\hat{x}_\mu^h$  can be constructed from the operator (2.7) as  $\hat{x}_\mu^h = \frac{1}{2}(\hat{x}_\mu + \hat{x}_\mu^\dagger) = (\hat{x}_\mu^h)^\dagger$ , which results in ( $\dagger$  here means the usual hermitian conjugation operation,  $x_\mu^\dagger = x_\mu$ ,  $\partial_\mu^\dagger = -\partial_\mu$ )

$$\hat{x}_\mu^h = x_\mu \sqrt{1 + a^2 p^2} - i M_{\mu\nu} a^\nu - i \frac{a^2}{2} \frac{1}{\sqrt{1 + a^2 p^2}} p_\mu. \quad (3.1)$$

The change of the specific realization forces us to replace the star product (2.12) with a new one corresponding to the hermitian realization of the  $\kappa$ -Minkowski spacetime [24]:

$$(f \star_h g)(x) = \lim_{\substack{u \rightarrow x \\ y \rightarrow x}} \mathcal{M} \left( e^{x^\mu (\Delta - \Delta_0) \partial_\mu} \sqrt[4]{\frac{1 - a^2 \Delta(\partial^2)}{(1 - a^2 \partial^2 \otimes 1)(1 - a^2 1 \otimes \partial^2)}} f(u) \otimes g(y) \right), \quad (3.2)$$

where it is understood that the coproduct  $\Delta(\partial_\mu)$ , Eq.(2.8), is a homomorphism, i.e.  $\Delta(\partial^2) = \Delta(\partial_\mu) \Delta(\partial^\mu)$ . In this way, the nonhermitian version of the star product (2.12) is replaced by the above hermitian one.

The  $\kappa$ -deformed star product  $\star_h$  (3.2) is associative in the same sense as the star product (2.12). However the star product  $\star_h$ , contrary to the star product (2.12), has the same trace and/or integral property as the usual Moyal-Weyl product:

$$\int d^n x \phi^\dagger \star_h \psi = \int d^n x \phi^* \cdot \psi, \quad (3.3)$$

where the asterisk  $*$  denotes usual complex conjugation, and  $\dagger$  means complex conjugation on the  $\kappa$ -Minkowski [24].

The above results – the new  $\star_h$ -product (3.2), and the identity (3.3) – embrace a very nice and important property: the integral measure problems are avoided due to the measure function absorbtion within the new,  $\kappa$ -deformed,  $\star_h$ -product (3.2). In the new action

$$\begin{aligned} S_n[\phi] = & \int d^n x (\partial_\mu \phi)^\dagger \star_h (\partial^\mu \phi) + m^2 \int d^n x \phi^\dagger \star_h \phi \\ & + \frac{\lambda}{4} \int d^n x \frac{1}{2} (\phi^\dagger \star_h \phi^\dagger \star_h \phi \star_h \phi + \phi^\dagger \star_h \phi \star_h \phi^\dagger \star_h \phi), \end{aligned} \quad (3.4)$$

the interaction  $\phi^4$  term should in fact incorporate six terms corresponding to all possible permutations of fields  $\phi$  and  $\phi^\dagger$ . However, due to the integral property (3.3) of the star

product (3.2), these six permutations can be reduced to only two mutually nonequivalent terms.

When expanded up to the first order in the deformation parameter  $a$ , the action (3.4), after rearrangements including integration by parts, receives the following form

$$\begin{aligned}
S_n[\phi] = & \int d^n x \left[ (\partial_\mu \phi^*)(\partial^\mu \phi) + m^2 \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 \right] \\
& + i \frac{\lambda}{4} \int d^n x \left[ a_\mu x^\mu \left( \phi^{*2} (\partial_\nu \phi) \partial^\nu \phi - \phi^2 (\partial_\nu \phi^*) \partial^\nu \phi^* \right) \right. \\
& \left. + a_\nu x^\mu \left( \phi^2 (\partial_\mu \phi^*) \partial^\nu \phi^* - \phi^{*2} (\partial_\mu \phi) \partial^\nu \phi \right) + \frac{1}{2} a_\nu x^\mu \phi^* \phi \left( (\partial_\mu \phi^*) \partial^\nu \phi - (\partial_\mu \phi) \partial^\nu \phi^* \right) \right].
\end{aligned} \tag{3.5}$$

At the end of this subsection note that the Hopf algebra, yielding (3.4), is a twisted symmetry algebra, where existence/conservation of charges and currents are still subject of research. However the action (3.5), obtained by expansion of (3.4) up to the first order in the deformation parameter  $a$ , is invariant under the internal symmetry transformations.

### 3.2 Equations of motion and Noether currents of internal symmetry

We proceed further by determining the equations of motion for the fields  $\phi$  and  $\phi^*$ :

$$\begin{aligned}
(\partial_\mu \partial^\mu - m^2) \phi = & \frac{\lambda}{6} \left[ \phi^* \phi^2 + i a_\mu x^\mu \left( \phi^2 \partial_\nu \partial^\nu \phi^* + \phi^* (\partial_\nu \phi) \partial^\nu \phi + \phi (\partial_\nu \phi) \partial^\nu \phi^* \right) \right. \\
& - i a^\mu x^\nu \left( \phi^* (\partial_\nu \phi) \partial_\mu \phi + \phi (\partial_\nu \phi) \partial_\mu \phi^* + \phi^2 \partial_\mu \partial_\nu \phi^* + \phi (\partial_\mu \phi) \partial_\nu \phi^* \right) \\
& \left. + \frac{i}{4} (1-n) a^\mu \phi^* \phi \partial_\mu \phi + \frac{i}{2} (1-n) a^\mu \phi^2 \partial_\mu \phi^* \right],
\end{aligned} \tag{3.6}$$

where  $\phi = 0$  is the trivial solution of the above equation, as it should be. The equation of motion for  $\phi^*$  can be obtained from (3.6).

Next, we present Noether currents derived from the Lagrangian densities (3.5):

$$\begin{aligned}
j^\mu(x) = & i \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \phi - i \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^*)} \phi^*, \\
\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} = & \frac{1}{2} \partial^\mu \phi^* + \frac{i\lambda}{4} \left[ \phi^{*2} (2a_\nu x^\nu \partial^\mu - a^\nu x^\mu \partial_\nu - a^\mu x^\nu \partial_\nu) \phi + \frac{1}{2} \phi^* \phi (a^\mu x^\nu \partial_\nu - a^\nu x^\mu \partial_\nu) \phi^* \right], \\
\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^*)} = & \frac{1}{2} \partial^\mu \phi - \frac{i\lambda}{4} \left[ \phi^2 (2a_\nu x^\nu \partial^\mu - a^\nu x^\mu \partial_\nu - a^\mu x^\nu \partial_\nu) \phi^* + \frac{1}{2} \phi^* \phi (a^\mu x^\nu \partial_\nu - a^\nu x^\mu \partial_\nu) \phi \right].
\end{aligned} \tag{3.7}$$

The above current (3.7), is conserved; that is  $\partial_\mu j^\mu(x) = 0$ , as it should be due to the invariance of the Lagrangian (3.5) under the internal symmetry transformations. This can be showed by straightforward computations from (3.4)-(3.7).

## 4. Perturbative study of the model two-point function

### 4.1 Feynman rules

Even though the S-matrix LSZ formalism, including Wick theorem, is not quite clearly defined on  $\kappa$ -Minkowski noncommutative spacetime, we continue bona fide towards the computation of the quantum properties of the model described by the action (3.5).

Due to the  $\kappa$ -deformation of our model, the statistics of particles is twisted, so that we are generally no more dealing with pure bosons. We are in fact dealing with *something* whose statistics is governed by the statistics flip operator [40, 41, 42] and the quasitriangular structure (universal R-matrix) on the corresponding quantum group [38, 43, 44, 45]. It would be interesting to investigate these mutual relations more thoroughly, but at the surface level, we can argue that it is possible to pick up the basic characteristics of the twisted statistics by using the nonabelian momentum addition law [17, 18, 46, 47]. It can be seen that the accordingly induced deformation of the  $\delta$ -function (arising from the implementation of the nonabelian momentum addition/subtraction rule) yields the usual  $\delta$ -function multiplied by a certain statistical factor. When we speak about deformed statistics, we have in mind a less rigid notion of statistics as applied to the symmetry properties of the states, where multiparticle change of the 4-momenta may change the state's symmetry properties.

In order to obtain the Feynman rules in momentum space, we are suggesting to use the following line of reasoning, which we shall further on call *hybrid* approach.

(A) We use the methods of standard QFT and treat the modifications in action (3.5) as a perturbation. In doing this, we obtain propagators and Feynman rules for vertices.

(B) We know that the statistics of particles is twisted and that it has to be implemented into the formalism. Thus, we require that the ordinary addition/subtraction rule induces deformed rule for twisted statistics on the  $\kappa$ -Minkowski spacetime. This, in momentum space, means

$$\sum_i k_i^\mu \rightarrow \sum_{\oplus i} k_i^\mu \quad \& \quad \sum_i k_i^\mu - \sum_j p_j^\mu \rightarrow \sum_{\oplus i} k_i^\mu \ominus \sum_{\oplus j} p_j^\mu, \quad (4.1)$$

where induced *deformed addition/subtraction* rules are going to be defined in Subsection 4.1.2, for the simplest cases of two to four momenta. The associativity of the direct sum  $\oplus$  is satisfied due to the associativity of the star product (3.2). We proceed in two steps:

- (1) Following above arguments we implement the induced conservation law within the delta functions in the Feynman rule, and
- (2) whenever needed, we use the modified/deformed conservation law along the course of evaluation of the Feynman diagrams.

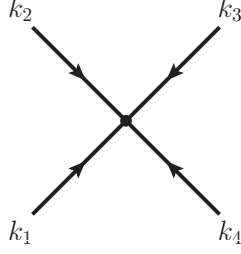
#### 4.1.1 Feynman rules (A): standard momentum addition law

From now on we continue to work with Euclidean metric. In transition from Minkowski to Euclidean signature we are using the transition rules:  $a^M = (a_0^M, a_i^M) \longrightarrow a^E = (a_i^E, a_n^E)$ , where  $a_n^E = ia_0^M$ , and similary for any  $n$ -vector. Thus the scalar product is defined as  $a^E k^E = a_\mu^E k_\mu^E = a_i^E k_i^E + a_n^E k_n^E = -a_0^M k_0^M + a_i^M k_i^M = a^M k^M$ . In subsequent consideration we drop the  $M/E$  superscripts, but it is understood that we work with Euclidean quantities.

In the further computation we are using the following full propagator

$$G \equiv G(k_1, k_2) = \frac{i}{k_1^2 + m^2} \delta^{(n)}(k_1 - k_2). \quad (4.2)$$





**Figure 1:** Scalar 4-field vertex

The vertex function, illustrated in Fig. 1, in the momentum space is given by

$$\tilde{\Gamma}(k_1, k_2, k_3, k_4; a) = i \frac{\delta^4 S[\tilde{\phi}]}{\delta \tilde{\phi}(k_1) \delta \tilde{\phi}(k_2) \delta \tilde{\phi}^*(k_3) \delta \tilde{\phi}^*(k_4)}, \quad (4.3)$$

and amounts to the following expression:

$$\begin{aligned} \tilde{\Gamma}(k_1, k_2, k_3, k_4; a) = & i(2\pi)^n \frac{\lambda}{2} a_\nu \left[ \frac{a_\nu}{a^2} + \frac{1}{4} \left( k_{4\mu} k_{3\nu} + k_{3\mu} k_{4\nu} - 2\delta_{\mu\nu} k_{4\rho} k_{3\rho} \right. \right. \\ & + \frac{1}{2} (k_{2\mu} k_{4\nu} - k_{4\mu} k_{2\nu} + k_{2\mu} k_{3\nu} - k_{3\mu} k_{2\nu}) \left. \right) \partial_\mu^{k_1} \\ & + \frac{1}{4} \left( k_{4\mu} k_{3\nu} + k_{3\mu} k_{4\nu} - 2\delta_{\mu\nu} k_{4\rho} k_{3\rho} \right. \\ & + \frac{1}{2} (k_{1\mu} k_{4\nu} - k_{4\mu} k_{1\nu} + k_{1\mu} k_{3\nu} - k_{3\mu} k_{1\nu}) \left. \right) \partial_\mu^{k_2} \\ & \left. + \frac{1}{4} \left( k_{1\mu} k_{2\nu} + k_{2\mu} k_{1\nu} - 2\delta_{\mu\nu} k_{1\rho} k_{2\rho} \right) (\partial_\mu^{k_3} + \partial_\mu^{k_4}) \right] \delta^{(n)}(k_1 + k_2 - k_3 - k_4), \quad (4.4) \end{aligned}$$

where we denote  $\partial_\mu^k = \frac{\partial}{\partial k_\mu}$ , and all four momenta  $k_i$  are flowing into the vertex. The coupling  $\lambda$  has to be dimensionally regularized.

#### 4.1.2 Feynman rules (B): $\kappa$ -deformed momentum addition law

Next, we discuss the notion which anticipates the induced momentum conservation law on the  $\kappa$ -space, within our *hybrid* approach. Namely, the  $\delta$ -function in (4.4) comes from the contraction of fields, where the momentum conservation should be obeyed in accordance with the  $\kappa$ -deformed momentum addition rule [17]. We have two cases for summation/subtraction of 4-vectors  $k^\mu$  with respect to the physical situation of four particles and/or quantum fields propagating in space with respect to an interaction point:

(I) all particle momenta flowing into the vertex, as given in Fig. 1

$$k_{1\mu} + k_{2\mu} + k_{3\mu} + k_{4\mu} = 0 \quad \rightarrow \quad k_{1\mu} \oplus k_{2\mu} \oplus k_{3\mu} \oplus k_{4\mu} = 0, \quad (4.5)$$

(II) the process of scattering "2 particle  $\rightarrow$  2 particle", where we have

$$(k_{1\mu} + k_{2\mu}) - (k_{3\mu} + k_{4\mu}) = 0 \quad \rightarrow \quad (k_{1\mu} \oplus k_{2\mu}) \ominus (k_{3\mu} \oplus k_{4\mu}) = 0. \quad (4.6)$$

Having defined the Feynman rules (4.2) and (4.4), we have completed the first stage of our program, that is to deduce the free propagation and interaction properties of the model by using the standard quantization.

At this point we turn to the second part, which includes the effective description of the statistics of particles described by the model. As already indicated before, the statistics of particles is twisted in  $\kappa$ -space [4, 14], with the deformation being encoded in the nonabelian momentum addition rule. It is known that the rule for addition of momenta is governed by the coproduct structure of the Hopf algebra in question. In our case, the relevant Hopf algebra is the  $\kappa$ -Poincaré algebra and the corresponding coalgebra structure is given by (2.8), (2.9), and (2.10), (2.11). In particular, the coproduct (2.8) for translation generators determines the required momentum addition rule, which in the momentum space and up to the first order in deformation  $a$ , from (2.10) and by the expansion  $\Delta\partial_\mu = \partial_\mu \otimes \mathbf{1} + \mathbf{1} \otimes \partial_\mu - i\partial_\mu \otimes a\partial + ia_\mu\partial_\alpha \otimes \partial^\alpha - \frac{1}{2}a^2\partial_\mu \otimes \partial^2 - a_\mu(a\partial)\partial_\alpha \otimes \partial^\alpha + \frac{1}{2}a_\mu\partial^2 \otimes a\partial + \mathcal{O}(a^3)$ , yields:

$$S(p_\mu) = -iS(\partial_\mu) = -p_\mu - a_\mu p^2 + (ap)p_\mu + \mathcal{O}(a^2), \quad (4.7)$$

$$(p_\mu \oplus k_\mu) = (p + k)_\mu + (ak)p_\mu - a_\mu(pk) + \mathcal{O}(a^2). \quad (4.8)$$

Here we have the nonabelian momentum addition rule (4.8), while  $S(p)$  is the antipode with the property  $p^\mu \oplus S(p^\mu) = 0$ , which in fact represents the very definition of the antipode. Namely, since commutativity in momentum space is not satisfied, i.e.  $k \oplus p \neq p \oplus k$ , a certain ordering has to be implemented. However, instead of implementation of a possibly complicated unknown ordering, we shall proceed in the most simple way by taking into account all possible types of contributions. This is symmetric ordering; for example  $k \oplus p \oplus q$ ,  $p \oplus k \oplus q$ , etc. Combining (4.7) and (4.8) we obtain the following momentum subtraction rule:

$$p_\mu \ominus k_\mu \equiv (p \oplus S(k))_\mu = (p - k)_\mu(1 - ak) + a_\mu(pk - k^2) + \mathcal{O}(a^2). \quad (4.9)$$

This enables us to rewrite the energy-momentum conservation which is assumed to be satisfied at each vertex. Thus, if two external momenta  $k_1$  and  $k_2$  flow into the vertex and the other two external momenta  $k_3$  and  $k_4$  flow out of the vertex, then, written in components, we have the induced momentum conservation law (4.6).

In order to obtain the expressions for the  $\delta$ -functions in the Feynman rules, we start with

$$\delta^{(n)}(p \ominus k) = \sum_i \left| \det \left( \frac{\partial(p \ominus k)_\mu}{\partial p_\nu} \right) \right|_{p=q_i}^{-1} \delta^{(n)}(p - q_i), \quad (4.10)$$

where we have to sum over all zeros  $q_i$  for the expression in the argument of the  $\delta$ -function on the LHS. Since there is only one zero,  $q_i = k$ , with the help of subtraction rule (4.9), we find the following first order contribution to the above  $\delta^{(n)}$ -function

$$\delta^{(n)}(p \ominus k) = \frac{\delta^{(n)}(p - k)}{(1 - ap)^{n-1}} = (1 + (n-1)ap + \mathcal{O}(a^2))\delta^{(n)}(p - k). \quad (4.11)$$

It was shown in [24] that the star product  $\star_h$  (3.2) breaks translation invariance (in the sense of the definition introduced in [48]). However, this feature does not show up until the computations are extended to second order in the deformation parameter  $a$ . The important point is that translation invariance is intact to first order. Since we are carrying out our study in exactly this order, we are allowed to invoke the energy momentum conservation albeit in a modified form, dictated by the modified coproduct structure. The relation between Hopf algebra symmetries and conservation laws is an important subject of investigation. This is the issue of generalizing the Noether theorem [49].

With the idea of implementing the new physical features that have just been described, we modify the Feynman rule (4.4). With the help of (4.8), (4.9), and, in the spirit of our *hybrid* approach [31, 50], by choosing the following replacement of the  $\delta$ -function in (4.4)

$$\delta^{(n)}(k_1 + k_2 - k_3 - k_4) \rightarrow \delta^{(n)}((k_1 \oplus k_2) \ominus (k_3 \oplus k_4)) + \delta^{(n)}((k_1 \oplus k_2) \ominus (k_4 \oplus k_3)), \quad (4.12)$$

we obtain the *hybrid* Feynman rule which obeys the  $\kappa$ -deformed momentum addition/subtraction rule using the sum of the  $\delta$ -functions (4.12):

$$\begin{aligned} \tilde{\Gamma}(k_1, k_2, k_3, k_4; a) = & i(2\pi)^n \frac{\lambda}{2} a_\nu \left[ \frac{a_\nu}{a^2} + \frac{1}{4} \left( k_{4\mu} k_{3\nu} + k_{3\mu} k_{4\nu} - 2\delta_{\mu\nu} k_{4\rho} k_{3\rho} \right. \right. \\ & + \frac{1}{2} (k_{2\mu} k_{4\nu} - k_{4\mu} k_{2\nu} + k_{2\mu} k_{3\nu} - k_{3\mu} k_{2\nu}) \left. \right) \partial_\mu^{k_1} \\ & + \frac{1}{4} \left( k_{4\mu} k_{3\nu} + k_{3\mu} k_{4\nu} - 2\delta_{\mu\nu} k_{4\rho} k_{3\rho} \right. \\ & + \frac{1}{2} (k_{1\mu} k_{4\nu} - k_{4\mu} k_{1\nu} + k_{1\mu} k_{3\nu} - k_{3\mu} k_{1\nu}) \left. \right) \partial_\mu^{k_2} \\ & + \frac{1}{4} \left( k_{1\mu} k_{2\nu} + k_{2\mu} k_{1\nu} - 2\delta_{\mu\nu} k_{1\rho} k_{2\rho} \right) (\partial_\mu^{k_3} + \partial_\mu^{k_4}) \left. \right] \\ & \times \left[ \delta^{(n)}((k_1 \oplus k_2) \ominus (k_3 \oplus k_4)) + \delta^{(n)}((k_1 \oplus k_2) \ominus (k_4 \oplus k_3)) \right]. \end{aligned} \quad (4.13)$$

In the above, the sum of the  $\delta$ -functions represents all mutually different physical situations.

The  $\delta$ -functions in (4.13), should in principle come from the contraction of fields quite naturally, if the noncommutative version of the LSZ formalism is applied to our model. Since such a formalism has not been developed so far, we choose to follow a kind of *hybrid* approach that combines the standard quantum field theory consideration with the peculiarities resulting from the statistics properties of particles in  $\kappa$ -space. The latter part is realized through embedding a nonabelian momentum-energy conservation within the 4-point vertex function. In this sense, *the hybrid* approach can serve as an intermediate step bridging the gap between the standard quantum field theory and the complete field theory on  $\kappa$ -space.

The Feynman rule (4.13) appears to be consistent with the energy-momentum conservation that respects  $\kappa$ -deformed momentum addition rule. In order to obtain the complete expression for the  $\delta$ -functions appearing in (4.13), we are proceeding in two steps. With

the help of (4.8)/(4.11), up to linear order in  $a$ , and with  $j, l = 3, 4; j \neq l$ , we have:

$$\begin{aligned}
\delta^{(n)}((k_1 \oplus k_2) \ominus (k_j \oplus k_l)) &= \frac{\delta^n((k_1 \oplus k_2) - (k_j \oplus k_l))}{\left(1 - a(k_1 \oplus k_2)\right)^{n-1}} \\
&= \frac{\delta^{(n)}((k_1 \oplus k_2) - (k_j \oplus k_l))}{\left[1 - \left(a(k_1 + k_2) + (ak_1)(ak_2) - a^2(k_1 k_2) + \mathcal{O}(a^3)\right)\right]^{n-1}} \\
&= (1 + (n-1)a(k_1 + k_2) + \mathcal{O}(a^2))\delta^{(n)}((k_1 \oplus k_2) - (k_j \oplus k_l)).
\end{aligned} \tag{4.14}$$

The second step is to compute the delta functions from (4.14) in the same way as in (4.11):

$$\delta^{(n)}((k_1 \oplus k_2) - (k_j \oplus k_l)) = \sum_i \left| \det \left( \frac{\partial((k_1 \oplus k_2) - (k_j \oplus k_l))_\mu}{\partial k_{1\nu}} \right) \right|_{k_1=q_i}^{-1} \delta^{(n)}(k_1 - q_i), \tag{4.15}$$

where we have to sum up over all zeros  $q_i$  for the expression in the argument of the  $\delta$ -function.

Next, we shall choose the specific momenta  $k_2 = k_3 = \ell$  we need for the evaluation of the tadpole diagram. Because there are no zeros for the delta function  $\delta^{(n)}((k_1 \oplus \ell) - (\ell \oplus k_4))$ , the only contribution comes from the second combination  $\delta^{(n)}((k_1 \oplus \ell) - (k_4 \oplus \ell))$ . In order to perform that computation, we start with (4.8) and orient the vector  $a$  in the direction of time,  $a = (0, \dots, 0, ia_0)$ . Due to covariance, the obtained result will also be valid for an arbitrary orientation of  $a$ . Hence

$$\begin{aligned}
\det \left( \frac{\partial((k_1 \oplus \ell) - (k_4 \oplus \ell))_\mu}{\partial k_{1\nu}} \right)_{k_1=k_4} &= \begin{vmatrix} 1 & -ia_0\ell_1 & \cdots & -ia_0\ell_{n-2} & -ia_0\ell_{n-1} \\ 0 & 1 + a\ell & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 + a\ell & 0 \\ 0 & 0 & \cdots & 0 & 1 + a\ell \end{vmatrix} \\
&= (1 + a\ell)^{n-1}.
\end{aligned} \tag{4.16}$$

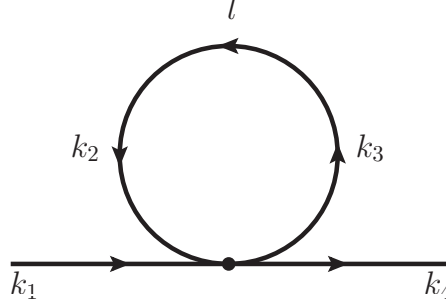
Since there is only one zero,  $q_i = k_4$ , we find

$$\delta^{(n)}((k_1 \oplus \ell) - (k_4 \oplus \ell)) = \frac{\delta^{(n)}(k_1 - k_4)}{(1 + a\ell)^{n-1}} = \left(1 - (n-1)a\ell\right)\delta^{(n)}(k_1 - k_4) + \mathcal{O}(a^2), \tag{4.17}$$

which gives the final expression

$$\begin{aligned}
\delta^{(n)}((k_1 \oplus \ell) \ominus (k_4 \oplus \ell)) &= \left(1 + (n-1)a(k_1 + \ell)\right) \frac{\delta^{(n)}(k_1 - k_4)}{(1 + a\ell)^{n-1}} + \mathcal{O}(a^2) \\
&= (1 + (n-1)ak_1)\delta^{(n)}(k_1 - k_4) + \mathcal{O}(a^2).
\end{aligned} \tag{4.18}$$

In the above expression the  $\ell$  dependences drop out as we expected, thus showing the consistency of the *hybrid* Feynman rule derivation. The remaining factor  $(1 + (n-1)ak_1)$  in (4.18) is due to the  $\kappa$ -space twisted particle statistics of our *hybrid* approach.



**Figure 2:** Scalar 4-field tadpole

## 4.2 Scalar field propagation

### 4.2.1 Tadpole diagram: standard momentum conservation

In order to compute the tadpole diagram from Fig. 2, using dimensional regularization, we have to introduce in the action (3.5) new dimensionful regularization masses denoted by  $\mu$  for the coupling  $\lambda$ . In accordance with QFT [51], the regularization of the  $\phi^4$  model requires:

$$\lambda_{new} = \lambda_{old}(\mu^2)^{\frac{n}{2}-2} \rightarrow (\mu^2)^{2-\frac{n}{2}}\lambda, \quad \lambda = \lambda_{new}. \quad (4.19)$$

We shall further restrict the computation only to the first order contribution of the two-point function  $\Pi_2^a$  in our model (3.5) corresponding to the diagram from Fig. 2:

$$\Pi_2^a = \Pi_2^0 + \Pi_2^{a \neq 0} = \int \frac{d^n k_2}{(2\pi)^n} \frac{d^n k_3}{(2\pi)^n} \tilde{\Gamma}(k_1, k_2, k_3, k_4; a, \mu) G(k_2, k_3; \mu). \quad (4.20)$$

In general, the one-loop integral (4.20) produces nonzero contributions, where  $G(k_2, k_3; \mu)$  is given by (4.2). Assuming momentum conservation,  $k_1 + k_2 = k_3 + k_4$  and  $k_2 = k_3 = \ell$  we have

$$\Pi_2^a = \int \frac{d^n \ell}{(2\pi)^n} \tilde{\Gamma}(k_1, \ell, \ell, k_4; a, \mu) \frac{i}{\ell^2 + m^2}. \quad (4.21)$$

First we need  $\tilde{\Gamma}$  from the Feynman rule (4.4) in accordance with the notations of Fig. 1; that is we have to replace the incoming momenta  $k_1$  and outgoing momenta  $k_4$  by  $k_3 \rightarrow -k_3 = -\ell$  and  $k_4 \rightarrow -k_4$ . Thus, we have

$$\begin{aligned} \tilde{\Gamma}(k_1, \ell, \ell, k_4^{out}; a, \mu) &= i(2\pi)^n \mu^{4-n} \frac{\lambda}{2} \left\{ 1 + a_\nu \frac{1}{8} \left[ \left( k_{4\mu} k_{1\nu} - k_{1\mu} k_{4\nu} \right) \partial_\mu^\ell \right. \right. \\ &\quad - 2 \left( \ell_\mu k_{4\nu} - 3\ell_\nu k_{4\mu} + 8\delta_{\mu\nu} k_{4\rho} \ell_\rho \right) \partial_\mu^{k_1} - 2 \left( \ell_\mu k_{1\nu} + k_{1\mu} \ell_\nu - 2\delta_{\mu\nu} k_{1\rho} \ell_\rho \right) \partial_\mu^{k_4} \\ &\quad \left. \left. + \left( \ell_\mu (2k_4 - k_1)_\nu + \ell_\nu (2k_4 - 3k_1)_\mu - 4\delta_{\mu\nu} (k_4 - k_1)^\rho \ell_\rho \right) \partial_\mu^\ell \right] \right\} \delta^{(n)}(k_1 - k_4). \end{aligned} \quad (4.22)$$

In order to obtain  $\tilde{\Gamma}(k_1, \ell, \ell, k_4^{in}; a, \mu)$  we just have to replace  $k_4 \rightarrow -k_4$  in (4.22).

As a next step, we compute  $\Pi_2^a$  straightforwardly with the help of the notion that the integral (4.21) is an effective action describing the given one-loop quantum process. So, employing integration by parts in (4.21) and using dimensional regularization, we obtain

$$\Pi_2^0 = -\frac{\lambda}{2} I_0, \quad (4.23)$$

$$\Pi_2^{a \neq 0} = -\frac{\lambda}{2} \left\{ \frac{3}{8} (aK) I_0 - \frac{1}{4} (aK) \left( \delta_{\mu\nu} - \frac{a_\mu K_\nu}{(aK)} \right) I_{2,\mu\nu} \right\}, \quad (4.24)$$

where  $K = 2k_4 - k_1$ . In the above equations the presence of  $(2\pi)^n \delta^{(n)}(k_1 - k_4)$  is understood, although not being explicitly stated. For  $n = 4 - \epsilon$ , we have the well known integrals

$$\begin{aligned} I_0 &= (\mu^2)^{2-\frac{n}{2}} \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{\ell^2 + m^2} = \frac{m^2}{(4\pi)^2} \left[ \left( \frac{4\pi\mu^2}{m^2} \right)^{\frac{\epsilon}{2}} \Gamma(-1 + \frac{\epsilon}{2}) \right]_{\epsilon \rightarrow 0} \\ &= \frac{-1}{8\pi^2} m^2 \left[ \frac{1}{\epsilon} + \frac{\psi(2)}{2} + \log \sqrt{\frac{4\pi\mu^2}{m^2}} + \dots \right], \end{aligned} \quad (4.25)$$

$$I_{2,\mu\nu} = (\mu^2)^{2-\frac{n}{2}} \int \frac{d^n \ell}{(2\pi)^n} \frac{\ell_\mu \ell_\nu}{(\ell^2 + m^2)^2} = \frac{1}{2} \delta_{\mu\nu} I_0, \quad \delta_{\mu\nu} \delta_{\mu\nu} = n, \quad (4.26)$$

with a simple pole at  $\epsilon = 0$ . Thus the expression (4.25) is divergent in the UV cut-off.

The non-vanishing contributions come from the commutative parts, that is from (4.23). The integrals (4.25) and (4.26) for  $n = 4$  give  $\Pi_2^{a \neq 0} = 0$ , producing the very well known commutative result (4.23). All contributions proportional to  $a$ , coming from  $\kappa$ -Minkowski NC  $\phi^4$  theory cancel out, as one would naively expect by inspecting vertex (4.22).

Clearly, the one loop computation has to be modified by anticipating the momentum conservation on the  $\kappa$ -space. To illustrate that something nonstandard appears in our model (4.4), we start with the general one-loop integral (4.20). It should be noted that one cannot integrate over  $k_3$  using the first delta  $\delta^{(n)}(k_2 - k_3)$  – from the propagator – and replace  $k_3$  by  $k_2$  in the above expression as it stands, because of the derivative with respect to  $k_2$ . So, as a first step of computation we are using a simple trick:

$$\begin{aligned} \partial_\mu^{k_1} \delta^{(n)}(k_1 + k_2 - k_3 - k_4) &= \partial_\mu^{k_2} \delta^{(n)}(k_1 + k_2 - k_3 - k_4), \\ \partial_\mu^{k_1} \delta^{(n)}(k_1 + k_2 - k_3 - k_4) &= -\partial_\mu^{k_3} \delta^{(n)}(k_1 + k_2 - k_3 - k_4), \text{ etc.}, \end{aligned} \quad (4.27)$$

and than we rewrite (4.4) and (4.2) as follows:

$$\begin{aligned} \tilde{\Gamma}(k_1, k_2, k_3, k_4; a) G(k_2, k_3) &= \\ &\left\{ i(2\pi)^n \frac{\lambda}{4} \left[ 1 + a_\nu \left( 2(-k_{1\mu} k_{2\nu} - k_{2\mu} k_{1\nu} + k_{3\mu} k_{4\nu} + k_{4\mu} k_{3\nu} + 2\delta_{\mu\nu}(k_{1\rho} k_{2\rho} - k_{3\rho} k_{4\rho})) \right. \right. \right. \\ &\quad \left. \left. + \frac{1}{2}(k_{1\mu} k_{3\nu} - k_{3\mu} k_{1\nu} + k_{2\mu} k_{3\nu} - k_{3\mu} k_{2\nu} + k_{1\mu} k_{4\nu} - k_{4\mu} k_{1\nu} + k_{2\mu} k_{4\nu} - k_{4\mu} k_{2\nu}) \right) \partial_\mu^{k_1} \right] \\ &\quad \left. \times \delta^{(n)}(k_1 + k_2 - k_3 - k_4) \right\} \left[ \frac{i}{k_2^2 + m^2} \delta^{(n)}(k_2 - k_3) \right]. \end{aligned} \quad (4.28)$$

After performing the integration over  $k_3$  in (4.28) we obtain the following expression:

$$\Pi_2^{a \neq 0} = -\frac{\lambda}{8(2\pi)^n} ((ak_1)k_4 - (ak_4)k_1)_\mu \left[ \partial_\mu^{k_1} \delta^{(n)}(k_1 - k_4) \right] \int \frac{d^n k_2}{k_2^2 + m^2}, \quad (4.29)$$

where  $k_1$  and  $k_4$  are the external momenta. This expression is quadratically divergent in the UV cut-off representing the quantum loop modification of the free action (3.5), which can be nonzero because of the momentum conservation violation at the vertex. Due to the results (4.29) we obviously stumbled across the momentum nonconservation. Such results seem to favor our *hybrid* approach.

#### 4.2.2 Tadpole diagram: $\kappa$ -deformed momentum conservation

In the following computation of the tadpole diagram from Fig. 2, we fully implement the *hybrid* approach, that is the notion that standard momentum conservation is not satisfied. However, in accordance with our *hybrid* approach, at the end the undeformed momentum conservation law has to be applied.

In the next step, we are applying integration by parts. This of course plays an essential role in our *hybrid* approach. Performing the computation of all terms in the one-loop tadpole integral (4.21) with the Feynman rule (4.13) and the delta function (4.18) and for an arbitrary number of dimensions  $n$ , we obtain the following first order result:

$$\Pi_2^a = \Pi_2^0 + \Pi_2^{a \neq 0} = -\frac{\lambda}{2} \left[ (1 + (n-1)ak_1) + \frac{1}{2} \left(1 - \frac{n}{4}\right) (ak_1 - 2ak_4) \right] I_0. \quad (4.30)$$

The first term in (4.30) for  $n = 4$  corresponds to the result (4.23) from the previous subsection. From above formulas it is clear that there exist non-vanishing contributions even for  $n = 4$ . They are arising from the  $\kappa$ -deformed momentum conservation rule, and entering through the deformed  $\delta$ -function (4.18) in the *hybrid* Feynman rule (4.13).

For  $n = 4 - \epsilon$ , we obtain a modified expression for the tadpole in Fig. 2 in the limit  $\epsilon \rightarrow 0$ , where the  $1/\epsilon$  divergence is explicitly isolated. For conserved external momentum in accordance with (4.18), i.e. for  $k_1 = k_4 \equiv k$ , (4.25) and (4.30) finally lead to,

$$\Pi_2^a = \frac{\lambda m^2}{32\pi^2} \left[ (1 + 3ak) \left( \frac{2}{\epsilon} + \psi(2) + \log \frac{4\pi\mu^2}{m^2} \right) - \frac{9}{4} ak \right]. \quad (4.31)$$

The finite parts represent modifications of the scalar field self-energy and depend explicitly on the regularization parameter, and the mass of the scalar field, and it contains a correction  $ak$  due to the dependence on the energy  $|k|$ , where the actual scalar field self-energy modifications occur.

The result (4.31) is discussed next in the framework of Green's functions. Generally we know that by summing all the 1PI contributions, for the full free propagator (4.2), we get the following expression for the two-point connected Green's function [51]

$$G_{(c,2)}^a(k_1, k_4) = \left[ \frac{i}{k_1^2 + m^2} + \frac{i}{k_1^2 + m^2} \Pi_2^a \frac{i}{k_1^2 + m^2} + \dots \right], \quad (4.32)$$

and as an illustration we resum the above series into

$$G_{(c,2)}^a(k_1, k_4) \longrightarrow (2\pi)^n \delta^{(n)}(k_1 - k_4) \left[ \frac{i}{k_1^2 + m^2 - \Pi_2^a} \right]. \quad (4.33)$$

The genuine  $1/\epsilon$  divergence in  $\Pi_2^a$ , can only be removed by introducing the following counter term  $\delta m^2$ :

$$\delta m^2 \tilde{\phi}^*(k) \tilde{\phi}(k) = \frac{\lambda m^2}{32\pi^2} \left[ (1 + 3ak) \frac{2}{\epsilon} + f\left(\frac{4 - \epsilon}{2}, \frac{\mu^2}{m^2}, ak\right) \right] \tilde{\phi}^*(k) \tilde{\phi}(k), \quad (4.34)$$

where  $f$  is an arbitrary dimensionless function, fixed by renormalization conditions. Adding the above counterterm contribution to the previous expression (4.32) results in the shift  $m^2 \rightarrow m^2 + \delta m^2$  in (4.32)/(4.33), thus leading to

$$\begin{aligned} \tilde{G}_{(c,2)}^a(k_1, k_4) &= \left[ G_{(c,2)}^a(k_1, k_4) + G_{(c,2)}^a(k_1, k_4)(-\delta m^2)G_{(c,2)}^a(k_1, k_4) + \dots \right] \\ &\longrightarrow (2\pi)^n \delta^{(n)}(k_1 - k_4) \left[ \frac{i}{k_1^2 + m^2 + \delta m^2 - \Pi_2^a} \right], \end{aligned} \quad (4.35)$$

where  $\tilde{G}_{(c,2)}^a$  denotes the Green's function including the contribution from the counter term.

However, since  $\Pi_2^a$  was computed for the free propagator (4.2), it is consistent to compute the two-point connected Green's function under the same approximation. After the resummation of (4.32), using of (4.2), we find

$$G_{(c,2)}^a(k_1, k_4) = \frac{G}{1 - G \Pi_2^a}. \quad (4.36)$$

In order to identify the proper counter term for the above expression, we resum the series in (4.35) with the full free propagator  $i/(k_1^2 + m^2)$  and replace (4.32)  $\rightarrow$  (4.36):

$$\tilde{G}_{(c,2)}^a(k_1, k_4) = \frac{G}{1 + G(\delta m^2 - \Pi_2^a)}. \quad (4.37)$$

In (4.37),  $\delta m^2$  is a generic quantity. This is due to the fact that expression (4.31) contains the finite parts too. The requirement  $\delta m^2 = \Pi_2^a$  removes the infinity. Thus, we have

$$\tilde{G}_{(c,2)}^a(k_1, k_4) = \frac{G}{1 + G \frac{\lambda m^2}{32\pi^2} \left[ f - (1 + 3ak) \left( \psi(2) + \log \frac{4\pi\mu^2}{m^2} - \frac{9}{4} \frac{ak}{1+3ak} \right) \right]}. \quad (4.38)$$

Precise extraction and removal of the genuine UV divergence is performed next via (4.34) in the context of the discussion of  $\tilde{G}_{(c,2)}^a(k_1, k_4)$  for different energy regimes; that is from low energy to extremely high -Planck scale- energy propagation.

There is a very interesting property of expression (4.31) at extreme energies. Namely, there exists a term  $(1 + 3ak)$  which for  $(1 + 3ak \rightarrow 0)$  tends to zero linearly. For low energies and/or small  $\kappa$ -deformation  $a$ , i.e. for  $ak \simeq 0$ , (equivalent to  $a \simeq 0$ ), which is far away



from the point  $(1 + 3ak = 0)$ , this is not the case.

#### Low energy limit

Using the finite combination  $(\delta m^2 - \Pi_2^0)$  for low energy ( $ak = 0$ ), and at the order  $\lambda$

$$\delta m^2 - \Pi_2^0 = \frac{\lambda m^2}{32\pi^2} \left[ f - \psi(2) - \log \frac{4\pi\mu^2}{m^2} \right], \quad (4.39)$$

we get from (4.31) and (4.38):

$$\tilde{G}_{(c,2)}^0(k_1, k_4) = \frac{G}{1 + G \frac{\lambda m^2}{32\pi^2} \left( f - \psi(2) - \log \frac{4\pi\mu^2}{m^2} \right)}. \quad (4.40)$$

This expression has a pole in Minkowski space, and we can define the renormalization condition by requiring that the inverse propagator at the physical mass is equal to  $k_1^2 + m_{phys/low}^2$ . This choice determines uniquely the sum of the residual terms in (4.39), which is in accordance with the commutative  $\phi^4$  theory result [51].

#### Planckian energy limit

At the limiting point  $(1 + 3ak \rightarrow 0)$ , which corresponds to the extreme energies  $|k|$ , where the components of the  $\kappa$ -deformation parameter  $a_\mu$  are extremely small, of order Planck length, the divergence in (4.31) is removed *under the choice that  $(1 + 3ak)$  tends to zero linearly, i.e. with the same speed as  $\epsilon$  does*. That is, in the Planckian energy limit

$$\frac{(1 + 3ak) \rightarrow 0}{\epsilon \rightarrow 0} \longrightarrow \mathcal{O}(1), \quad (4.41)$$

the  $1/\epsilon$  and  $ak$  terms, from (4.31), do contribute.

Assuming that our  $\kappa$ -noncommutativity is spatial  $a_\mu = (\vec{a}, 0)$ , and using the momentum along the third axis  $k_\mu = (0, 0, E, iE)$ , i.e. for  $ak = Ea_3$ , Eq. (4.31) in the Planckian energy limit (4.41) gives

$$\Pi_2^a \Big|_{(3Ea_3+1 \rightarrow 0)} \longrightarrow \frac{\lambda m^2}{32\pi^2} \left[ 2 - \frac{9}{4}ak \right]_{(3Ea_3+1 \rightarrow 0)}, \quad (4.42)$$

producing the following modified Green's function (4.33):

$$\Pi_2^a \Big|_{(3Ea_3+1 \rightarrow 0)} \rightarrow \frac{\lambda m^2}{32\pi^2} \frac{11}{4} \Rightarrow \tilde{G}_{(c,2)}^a(k_1, k_4) \Big|_{(3Ea_3+1 \rightarrow 0)} \simeq \frac{i(2\pi)^4 \delta^{(4)}(k_1 - k_4)}{k_1^2 + m^2 \left( 1 - \frac{\lambda}{32\pi^2} \frac{11}{4} \right)}. \quad (4.43)$$

At the exact zero-point  $3Ea_3 + 1 = 0$  however, we obtain a different result from (4.31):

$$\Pi_2^a \Big|_{(3Ea_3+1=0)} = \frac{\lambda m^2}{32\pi^2} \frac{3}{4} \Rightarrow \tilde{G}_{(c,2)}^a(k_1, k_4) \Big|_{(3Ea_3+1=0)} \simeq \frac{i(2\pi)^4 \delta^{(4)}(k_1 - k_4)}{k_1^2 + m^2 \left( 1 - \frac{\lambda}{32\pi^2} \frac{3}{4} \right)}. \quad (4.44)$$

For the time-space noncommutativity with the time component of the  $\kappa$ -deformation parameter  $ia_0$  being also of the Planck length order and vanishing space component, i.e. for  $ak = -Ea_0$ , Eq. (4.31) in the Planckian energy limit (4.41) produces the result

$$\Pi_2^a \Big|_{(3Ea_0-1 \rightarrow 0)} \longrightarrow \frac{\lambda m^2}{32\pi^2} \left[ 2 - \frac{9}{4}ak \right]_{(3Ea_0-1 \rightarrow 0)}, \quad (4.45)$$

equivalent to (4.42), leading to the final results identical with (4.43) and (4.44).

The existence of the linear type of the limit ( $1 + 3ak \rightarrow 0$ ) which removes the genuine UV divergence  $1/\epsilon$  is a new, previously unknown feature of NC  $\kappa$ -Minkowski  $\phi^4$  theory at linear order in  $a$ . The above expressions give the  $\kappa$ -deformed dispersion relations. Mass shift receives the fixed value (4.42), independent of the function  $f$ , but it does depend on the scalar factor  $ak$ ; that is, it depends on the direction, on the energy  $|k|$  and on the  $\kappa$ -deformation parameter  $a$ . Thus, equations (4.42)/(4.45) and (4.43)/(4.44) represent a birefringence [52, 53] of the massive scalar field mode. Namely, both of the finite terms in (4.38) do not contribute in the limit  $(1 + 3ak) \rightarrow 0$ , and the possibility of their internal cancellation is diminished. The inverse propagator determines the physical mass  $m_{phys/Planck}^2$  at Planck scale energies.

## 5. Discussion and conclusion

A description and discussion of the paper's main results are in order.

- (i) The integral measure problems on  $\kappa$ -Minkowski spacetime are avoided by the introduction of the new  $\kappa$ -deformed  $\star_h$ -product (3.2), which naturally absorbs the measure function due to the hermitian realization (3.1).
- (ii) The trace/integral identity (3.3) is valid for  $\kappa$ -deformed spaces, but only if one is dealing with the hermitian realization (3.1), as one should, because only hermitian realizations have physical meaning. Due to the integral identity (3.3), the only deformation with respect to the standard scalar field action comes from the interaction term in (3.4).
- (iii) The action (3.5) produces modified equations of motion (3.6) and conserved deformed currents (3.7) due to the internal symmetry satisfied at that order.
- (iv) The truncated  $\kappa$ -deformed action (3.5) does not possess the celebrated UV/IR mixing [25]. The lack of the UV/IR mixing is a general feature of most of NCQFT expanded in terms of the noncommutative deformation parameter.
- (v) Next, we discuss the result for the tadpole diagram contribution to the propagation and/or self-energy of our scalar field  $\phi$  for arbitrary number of dimensions  $n$ , depicted in Fig. 2, as a function of the  $\kappa$ -deformed momentum conservation law. This originates from deformed statistics on the  $\kappa$ -Minkowski spacetime. Our approach is a kind of *hybrid* approach modeling between standard QFT and NCQFT on the  $\kappa$ -Minkowski spacetime, involving  $\kappa$ -deformed  $\delta$ -functions in the Feynman rules.
- (vi) When we worked with the standard conservation of momenta and the undeformed  $\delta$ -function, the contributions to the tadpole diagram to first order in  $a$  are zero. The deformed  $\delta$ -function can be written in terms of a leading term plus corrections in  $a$ . Since the Feynman rule (4.4) has already terms linear in  $a$ , we have to retain only the zeroth

order term in the modified  $\delta$ -function, because otherwise we get terms of quadratic and higher orders in  $a$  (and this is not what we are interested in). The only term where we need to take into account the  $\delta$ -function correction linear in  $a$ , is the leading order term in the Feynman rule (4.4). However, this term is vanishing due to integration over the loop momentum.

(vii) It appears that in the computation of the tadpole diagram integrals for arbitrary number of dimensions  $n$ , for  $n = 4$  all contributions linear in  $a$  canceled each other automatically. For dimensions  $n \neq 4$ , the same linear in  $a$  contributions become nonzero, regardless which momentum conservation rule is used. The propagation of the scalar field  $\phi$  for  $n = 4$  dimensions also receives a modification from the  $\kappa$ -deformation at linear order in the deformation parameter  $a$ .

(viii) In the final computation of the tadpole diagram depicted in Fig. 2, we fully implement the notion of our *hybrid* approach, i.e. we have to use the momentum conservation on  $\kappa$ -space given in (4.6), while at the end of the computation undeformed momentum conservation has to be applied. We have found non-vanishing contributions even in  $n = 4$  dimensions. They are arising from the  $\kappa$ -deformed momentum conservation rule entering through the deformed  $\delta$ -function in the *hybrid* Feynman rule (4.13). We have found a fully modified expression for the tadpole in Fig. 2 in the limit  $\epsilon \rightarrow 0$ , where the genuine  $1/\epsilon$  (UV) divergence is explicitly isolated. For conserved external momenta, i.e. for  $k_1 = k_4 \equiv k$ , we obtain the two-point function (4.31), where the finite parts represent the modification of the scalar field self-energy  $\Pi_2^g$  and depend explicitly on the regularization parameter  $\mu^2$  and the mass of the scalar field  $m^2$ . The most important is that (4.31) contains the finite correction  $ak$  due to the deformed statistics on the  $\kappa$ -Minkowski spacetime, thus, we obtain an explicit dependence on the direction of the propagating energy, its scale  $|k| = E$  and the  $\kappa$ -deformation parameter  $a$ , as we expected.

(ix) The two-point function (4.31) is next applied in the framework of the two-point connected Green's function for three energy regimes, that is for low energies, for Planck scale energies, and for intermediate energies, respectively. For low energy scale and/or small  $\kappa$ -deformation  $a$ , i.e. for  $ak \simeq 0$ , which is far away from the point  $(1 + 3ak = 0)$ , the  $\kappa$ -deformation dependence of the two-point Green's function completely drops out (4.40). The genuine UV divergence in (4.31) has been removed by subtracting the counterterm  $\delta m^2$  (4.34), from previous contribution (4.32), or through shifting  $m^2$  into  $(m^2 + \delta m^2)$  in (4.33). In this case the mass shift (4.40) could increase or decrease  $m^2$  depending on the values of the function  $f$ .

(x) For energies within the limits  $\frac{-1}{3a_3} \ll E \ll 0$  (or equivalently  $\frac{1}{3a_0} \gg E \gg 0$ ), the full expression (4.31), with the mass counterterm (4.34) has to be used in order to determine the Green's function (4.35).

(xi) At the Planckian energy scale, due to the existence of linear type of limits  $(1 + 3ak) \rightarrow 0$ , we have a new situation and distinguish two cases. They both are new, previously unknown features of the linear order in  $a$  NC  $\kappa$ -Minkowski  $\phi^4$  model. In the first case we have the limit (4.41) which produces the self-energy and/or the modified Green's function (4.43).

(xii) Second case, that is the exact zero-point where  $1 + 3Ea_3 = 0$ , represents in fact a genuine type of the zero-point which exactly removes the UV divergence  $1/\epsilon$ , producing

the self-energy and/or modified Green's function (4.44). In both cases the mass term is shifted in the same direction (*the same sign!*) but by a different amount,  $+11/4$  versus  $+3/4$ , respectively. Or more precisely, we can say that the mass shift during the limiting process ( $3Ea_3 + 1 \rightarrow 0$ ) drops from the value proportional to  $+11/4$  to the exact value proportional to  $+3/4$ .

(xiii) The results (4.43) and (4.44) are the same for two different choices of  $\kappa$ -noncommutativity, i.e. for choice  $a_\mu = (0, 0, a_3, 0)$ , or  $a_\mu = (0, 0, 0, ia_0)$ , respectively.

(xiv) At the Planckian propagation energy scale  $E \simeq \frac{-1}{3a_3}$ , the contribution of the tadpole in Fig. 2 tends to the fixed finite value, between (4.43) and (4.44), respectively. Due to effects of the  $\kappa$ -Minkowski statistics, this only depends on the direction of the propagation and the  $\kappa$ -deformation parameter  $a$ . In this way (4.42)/(4.45) and (4.43)/(4.44) represent  $\kappa$ -deformed dispersion relations, producing a genuine birefringence, [52, 53], of the massive scalar field modes, similarly to the chiral fermion field birefringence in the truncated model [53].

(xv) Considering full renormalization, besides the  $\delta m^2$  counterterm, also the other divergent parts have to be added as counterterms to the free Lagrangian (3.5):

$\int (\mathcal{L} + \mathcal{L}_{ct}) = S[\phi_B, m_B, \lambda_B, a_B]$ , where the index  $B$  denotes bare quantities. That would include an analysis of the 4-point one-loop contributions, counterterms  $(\mu^2)^{2-\frac{n}{2}}\delta\lambda$  as well as 2-loop expansion for the 2-point Green's function with insertion of the counterterms in multi-loop diagrams. Certainly, the full analysis of the renormalization group equations should be studied following the same lines. However, the full renormalization of our action (3.5) is anyhow beyond the scope of this paper and it is planned for our next project.

Regarding the effects of the statistics according to the described arguments, we repeat that within first order in the  $\kappa$ -deformation  $a$ , *the statistics effects on the  $\kappa$ -Minkowski in our hybrid model do arise as semiclassical/hybrid behavior of the first order quantum correction, thus showing birefringence of the massive scalar field mode.*

## Acknowledgments

We would like to acknowledge A. Andraši, A. Borowiec, H. Grosse, J. Lukierski, V. Radovanović and J. You for fruitful discussions. We would specially like to thank to J. Lukierski and A. Borowiec for careful reading of the manuscript and numerous valuable remarks which we incorporated into the final version of this manuscript. J.T. would like to acknowledge support of Alexander von Humboldt Foundation (KRO 1028995), and Max-Planck-Institute for Physics, and W. Hollik for hospitality. This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contract No. 098-0000000-2865 and 098-0982930-2900.

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